

# Regulation of Selective Harvesting of A Prey-Predator Fishery With Low Predator Density: A Capital Theoretic Analysis

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**ABSTRACT:** This paper deals with the selective harvesting of a prey-predator fishery with low predator density, the prey species obeying the logistic law of growth [1]. In this paper it is assumed that fishing is permitted only for the predator species. In order to control over exploitation of the predator species the regulatory agencies impose a suitable tax per unit harvested biomass of the landed fish. Here the effort is considered as a dynamic variable i.e. time dependent variable. The existence and their local stability of the equilibrium points of the dynamical system are discussed. The global stability of the non-trivial equilibrium points for both the exploited system and the unexploited system are discussed. The results are illustrated with the help of two numerical examples.

**KEYWORDS:** Selective harvesting, prey-predator fishery, low predator density, regulatory agencies, equilibrium points, local and global stability.

## I. INTRODUCTION

Prey-predator interaction is one of the common interactions between different species in nature. In the fish species, prey-predator interaction is a very common problem for extinction of the prey fish population. Moreover, common harvesting of prey-predator species is a global problem for extinction of both the species. So selective harvesting of a prey-predator fishery has an important role in the exploited fishery system. For existence of the biological equilibrium of both the prey and predator populations, the Government or private agencies have to monitor the harvest of the prey and predator species. It is better to restrict or stop the harvest of the prey population, if possible.

In order to control over exploitation of the species, the Government or the private agencies should regulate the harvesting effort. Various techniques in regulating fisheries have been discussed by Anderson and Lee [2], Sutinen and Anderson [3] and others. Taxation, license fees, lease of property rights, seasonal harvesting etc. are usually consider as possible governing instrument in fishery regulation. Among these, taxation is superior to other control policies because of its flexibility. As described by Clark ([4], Art.4.6, p-116), "Economists are particularly attracted to taxation, partly because of its flexibility and partly because many of the advantages of a competitive economic system can be better maintained under taxation than under other regulatory methods".

In most of the fishery models, the fishing effort is assumed to be constant. However, a fully dynamic, open access fishery model requires that the level of effort should expand or contract according as the perceived rent i.e. the net economic revenue of the fishermen is positive or negative. A model reflecting the dynamic interaction between the fishery and the effort is called a dynamic reaction model. Dynamic reaction models have so far been studied by Clark [5], Chaudhuri and Johnson [6], Ganguly and Chaudhuri [7] in the case of single species fishery models. Pradhan and Chaudhuri [8] studied a dynamic reaction model of a two species fishery with taxation as a control instrument. A dynamic reaction model of an inshore-offshore fishery with taxation as a control instrument is studied by Ray and Pradhan [9].

In this paper it is assumed that the growth rate of the prey species obeys the logistic law of growth and the predator population density is very low. It is also assumed that the regulatory agencies regulate the fishery by imposing a tax per unit harvested biomass for controlling over exploitation. Suitable ranges of tax are determined for which the non trivial

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steady states of the dynamical system exist. The fishing effort considered here is a dynamic variable depending on the net revenue to be earned by the fishermen. The existence of the steady states of the system is discussed. The local stability of all the steady states is discussed. The global stability of the non trivial steady states is studied for both the exploited and unexploited system. Two important theorems have been discussed about the non-trivial steady states for both the exploited and unexploited system. Two numerical examples are given to illustrate all the results and also for verification of the theorems.

## II. RELATED WORK

Prey-predator fishery model is an important population model and has been studied by many researchers. Saha Ray and Chaudhuri [10] developed a fishery model of two species Lotka-Volterra [11,12] prey-predator system where both the species are subjected to combined harvesting effort. This system was studied by taking into account time-periodic variations in biological parameters and harvesting effort. Bioeconomic exploitation of Lotka-Volterra prey-predator system with combined harvesting effort was studied by Chaudhuri and Saha Ray [13]. The optimal policy for the combined harvesting of a prey predator community was discussed by Mesterton-Giobbons [14]. Smith [15] developed a prey-predator fishery community with low predator density and Chaudhuri and Pradhan [16] extended this model considering combined harvesting effort. Kar [17] developed a prey-predator model with stage-structure for the predator species and selective harvesting of the prey species. Pradhan [18] discussed the dynamics of an exploited prey-predator fish species with Beddington-DeAngelis functional response [19,20]. The Beddington-DeAngelis functional response is similar to the Holling type II functional response but contains an extra term describing mutual interference by the predator. Srinivas et. al. [21] proposed a prey-predator fishery model with stage structure in two patchy marine aquatic environments. They studied the dynamics of a fishery resource system with stage structure in an aquatic environment that consists of two zones- unreserved zone where fishing is permitted and the reserved zone where fishing is prohibited.

Regulation of exploitation of fish populations has become a problem of major concern nowadays due to gradually decrease of resource stocks and the deteriorating environment. Taxation is one of the best regulatory methods to control over exploitation of the fish species. At first Clark [4] developed a single species fishery model using taxation as control instrument. Chaudhuri and Johnson [6] extended that model using a catch-rate function which was more realistic than that of used by Clark [4]. Pradhan and Chaudhuri [22] developed a mathematical model for growth and exploitation of a schooling fish species using a realistic catch-rate function and imposing a tax per unit biomass of the landed fish to control harvesting. Pradhan and Chaudhuri [8] developed a fully dynamic reaction model of open-access fishery consisting two competing fish species with taxation as a control instrument. In that model both the species obeyed the logistic law of growth and it was assumed that the gross rate of investment of capital in the fishery is proportional to the perceived rent (the net economic revenue to the fishermen). Kar and Chattopadhyay [23] developed a dynamic reaction model of a prey-predator system where the selective harvesting of prey species is considered. Ray and Pradhan [9] analyzed a mathematical model to study the dynamics of an inshore-offshore fishery with taxation as control measure. In that model the inshore area was considered as a breeding place having a fixed carrying capacity where fishing was strictly prohibited. Fishing was permitted only in the offshore area after imposing a suitable tax per unit biomass by the regulatory agencies in order to control over exploitation. But in reality it was very difficult to prevent the fishing in the inshore area entirely, because the low cost of fishing in the inshore area attracted all the fishermen to fish in that area. Moreover, the revenue earned by the Government or the regulatory agencies will be decreased if the fishing in the inshore area is totally stopped. Again, since the inshore area is the breeding place of the species so excess harvesting from that area may cause an over exploitation of the species. Considering all the above draw backs, Pradhan [24] developed a mathematical model with combined harvesting of an inshore-offshore fishery with taxation as a control instrument. In that paper inshore area is considered as the breeding place but fishing in that area is allowed after imposing a higher tax per unit harvested biomass from inshore area compared to the tax per unit harvested biomass from offshore area. This paper is a fully dynamic reaction model of a prey predator fishery with low predator density. It is also assumed that only the predator population is allowed to be harvested after imposing a suitable tax per unit harvested biomass in order to control over exploitation.

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## III. FORMULATION OF THE PROBLEM

Let us consider a prey-predator fishery in which the growth rate of the prey and predator populations obey the following dynamical system:

$$\left. \begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{k}\right) - \alpha xy \\ \frac{dy}{dt} &= \frac{\beta xy}{k} - sy \end{aligned} \right\} \quad (1)$$

where  $r, k, \alpha, \beta, s$  are all positive constants.

Here  $x(t)$  = the size or the biomass of the prey population at time  $t$ ,

$y(t)$  = the size or the biomass of the predator population at time  $t$ ,

$r$  = intrinsic growth rate of the prey species,

$\alpha$  = catchability rate i.e. the rate at which the prey species is removed due to predation effect,

$\beta$  = the maximum reproductive rate of the predator,

$s$  = per capita death rate of the predator and

$k$  = environmental carrying capacity.

Here the prey population obeys the logistic law of growth function  $rx \left(1 - \frac{x}{k}\right)$ . The term  $\frac{rx^2}{k}$  is called the crowding effect term which is present in the growth function of the prey population due to intra-specific competition, but absent in the growth equation of the predator population. This indicates that the predator density is low.

$$\text{Now, } \frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) \Rightarrow x(t) = \frac{k}{1 + \left(\frac{k-x_0}{x_0}\right)e^{-rt}}, x_0 = x(0).$$

This shows that in the absence of the predator, the prey population grows exponentially and the limiting value of the prey population is equal to the environmental carrying capacity. In the absence of the prey population, the predator population dies out exponentially with a rate  $s$ . The amount (or biomass) of the prey consumed by a predator in a unit time is  $\alpha x$  which may also be interpreted as the trophic function or the predator functional response to the prey population. This is also well known as Holling type-I functional response. A fraction  $\frac{\alpha}{\beta}$  of the energy consumed with this biomass goes into predator reproduction while the rest of the energy is spent to sustain metabolism and hunting activity of predators.

In spite of low predator density, it is assumed that only the predator species is allowed to be harvested by the fishermen due to higher market price of the predator fish compared to the prey fish. So the fishery will be profitable by harvesting only the predator fish population at a constant harvesting cost. Moreover, if the prey species is allowed to be harvested, then the growth rate of the predator will be declined due to limited resources and in future the predator species will be extinct. So the Government or any private agency should monitor the fishery for harvesting only the predator fish species. Again, in order to control the over exploitation of the predator species the Government or private agencies should impose a tax per unit biomass of the landed fish.

It is assumed that the predator species is subjected to harvesting effort  $E$ . So the second equation of the dynamical system (1) becomes  $\frac{dy}{dt} = \frac{\beta xy}{k} - sy - qEy$ , where  $q$  is the catchability coefficient of the predator species.

Here  $E$  is considered as a dynamic i.e. time dependent variable governed by the equations

$$E(t) = aQ(t), 0 \leq a \leq 1 \quad (2)$$

$$\text{and } \frac{dQ(t)}{dt} = I(t) - dQ(t), Q(0) = Q_0 \quad (3)$$

where  $I(t)$  is the gross investment rate at time  $t$ ,  $Q(t)$  is the amount of capital invested in the fishery at time  $t$  and  $d$  is the rate of depreciation of the capital. The fishing effort  $E$  at any time is assumed to be proportional to the instantaneous amount of the capital invested.

The regulatory agencies impose the tax  $\tau (> 0)$  per unit biomass of the landed fish to control the over exploitation of the predator species. The net economic revenue to the fishermen (perceived rent) is  $E\{q(p - \tau)y - c\}$ , where  $p$  is the market price of the predator fish which is assumed to be constant and  $c$  is the constant cost per unit of harvesting effort.

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It is assumed that the gross rate of investment of capital is proportional to the perceived rent.

Therefore,  $I(t) = b\{q(p - \tau)y - c\}E, 0 \leq b \leq 1$  (4)

This equation asserts that the maximum investment rate at any time equals the perceived rent (when  $b = 1$ ) at that time. The case  $b = 0$  may be used only in the situation of negative perceived rent, when the capital stocks are perfectly 'in-malleable' (i.e., when disinvestment of capital assets is not feasible). If the fishery operates on a loss and at the same time, the capital stocks are perfectly 'malleable' (i.e. disinvestment of capital is allowed), the sole owner of the fishery is benefited by allowing a continuous disinvestment of the capital assets. From the equations (2), (3) and (4), we have

$$\frac{dE}{dt} = [ab\{q(p - \tau)y - c\} - d]E \quad (5)$$

Therefore, we have the following system of equations:

$$\left. \begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{k}\right) - axy \\ \frac{dy}{dt} &= \frac{\beta xy}{k} - sy - qEy \\ \frac{dE}{dt} &= [ab\{q(p - \tau)y - c\} - d]E \end{aligned} \right\} \quad (6)$$

## IV. EXISTENCE OF EQUILIBRIUM POINTS

The equilibrium points of the system of equations (6) are given by  $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dE}{dt} = 0$ . (7)

Since  $x = y = E = 0$  always satisfy the equations (7), so  $P_0(0,0,0)$  is the trivial equilibrium point of the system of equations (6).

In absence of predator, effort will be zero and when  $y = E = 0$ , then  $\frac{dx}{dt} = 0 \Rightarrow x = 0, k$ .

Therefore,  $P_1(k, 0, 0)$  is the boundary or axial equilibrium point of the system of equations (6).

For the unexploited system ( $E = 0$ ), the non-trivial equilibrium point is given by

$$rx \left(1 - \frac{x}{k}\right) - axy = 0 \quad (8)$$

$$\text{and } \frac{\beta xy}{k} - sy = 0 \quad (9)$$

$$\text{From (9), we have } x = \frac{ks}{\beta} (> 0) = x_1 \text{ (say)} \quad (10)$$

$$\text{When } x = \frac{ks}{\beta}, \text{ then from (7), we have } y = \frac{r}{k} \left(1 - \frac{s}{\beta}\right) = y_1 \text{ (say)} \quad (11)$$

Now,  $y_1 > 0 \Rightarrow \beta > s$  i.e. the reproductive rate of the predator population is greater than the natural mortality rate of the predator population. In the ideal environmental conditions it is assumed that the reproductive rate is always greater than the death rate of the species and so throughout the paper it is assumed that  $\beta > s$ .

Therefore,  $P_1(x_1, y_1, 0)$  is the planar equilibrium point of the system (6).

$$\text{For the exploited system } (E \neq 0), \frac{dE}{dt} = 0 \text{ gives } y = \frac{abc+d}{abq(p-\tau)} = y^* \text{ (say),} \quad (12)$$

$$\frac{dx}{dt} = 0 \text{ gives } x = \frac{k}{r} (r - \alpha y) \text{ since } x \neq 0.$$

$$= k \left\{ 1 - \frac{\alpha(abc+d)}{abqr(p-\tau)} \right\} \text{ by (12)}$$

$$= x^* \text{ (say)} \quad (13)$$

$$\text{and } \frac{dy}{dt} = 0 \text{ gives } E = \frac{1}{q} \left( \frac{\beta x}{k} - s \right) \text{ since } y \neq 0$$

$$= \frac{1}{q} \left\{ \beta - s - \frac{\alpha\beta(abc+d)}{abqr(p-\tau)} \right\} = E^* \text{ (say)} \quad (14)$$

$$\text{Here } y^* = \frac{abc+d}{abq(p-\tau)} > 0 \text{ if and only if } 0 < \tau < p \quad (15)$$

i.e. the tax is less than the market price of the landed fish. Since the fishery is controlled by the regulatory agencies, so the agencies should impose the tax  $\tau$  such that  $0 < \tau < p$  for existence of the steady state of the predator species. It is seen that for the fixed harvesting cost, depreciation cost and the market price, the steady state level of the predator population depends only on the tax imposed by the regulatory agencies. If the regulatory agencies increase the tax, then the steady state level of the predator population also increase. In reality when the tax level is increased then the

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fishermen are unwilling to go to fishing and as a result the harvesting effort will be decreased and so the steady state level of the predator population will be increased. This result is shown graphically in the numerical section.

Now,  $x^* > 0$  if and only if  $\frac{\alpha(abc+d)}{abqr(p-\tau)} < 1$  by (13).

This implies  $0 < \tau < p - \frac{\alpha(abc+d)}{abqr}$  (16)

Again,  $E^* > 0$  if and only if  $\frac{\alpha\beta(abc+d)}{abqr(p-\tau)} < \beta - s$  by (14)

This implies  $0 < \tau < p - \frac{\alpha\beta(abc+d)}{abqr(\beta-s)}$  (17)

From (15), (16) and (17), we have

$$0 < \tau < \min\left\{p, p - \frac{\alpha(abc+d)}{abqr}, p - \frac{\alpha\beta(abc+d)}{abqr(\beta-s)}\right\} = p - \frac{\alpha\beta(abc+d)}{abqr(\beta-s)} \text{ since } \frac{\beta}{\beta-s} > 1.$$

From (12) and (13), we have  $x^* + y^* = \left\{k - \frac{k\alpha(abc+d)}{abqr(p-\tau)} + \frac{abc+d}{abq(p-\tau)}\right\}$ .

When  $\frac{r}{\alpha} = k$ , then  $x^* + y^* = k$ .

If the ratio of the intrinsic growth rate and the removal rate of the prey population is equal to the environmental carrying capacity, then the sum of the prey and predator populations remain constant which is equal to the carrying capacity of the environment.

Biologically it is true that the sum of the steady states of the prey and predator populations never exceeds the environmental carrying capacity  $k$ .

Therefore,  $x^* + y^* = \left\{k - \frac{abc+d}{abq(p-\tau)}\left(\frac{k\alpha}{r} - 1\right)\right\} < k$  implies  $\frac{r}{\alpha} < k$ .

Thus we have the following theorems:

## Theorem-1: (Conservation of population density)

Whatever the tax be imposed (within the suitable ranges), the sum of the prey and predator populations is always constant if and only if the ratio of the intrinsic growth rate of prey and the removal rate of the prey species by the predator species is equal to that constant. This constant is equal to the environmental carrying capacity. (Verification of this theorem is shown in numerical example-1)

## Theorem-2: (Non conservative prey predator system)

For the non conservative prey predator system, the ratio of the intrinsic growth rate of the prey species and the removal rate of the prey species by the predator species is always less than the environmental carrying capacity. (Verification of this theorem is shown in numerical example-2)

Therefore, the system of equations (6) has the following equilibrium points.

- i)  $P_0(0, 0, 0)$  is the trivial equilibrium point,
- ii)  $P_1(k, 0, 0)$  is the boundary or axial equilibrium point,
- iii)  $P_2(x_1, y_1, 0)$  is the non-trivial planar equilibrium point for the unexploited system where  $x_1$  and  $y_1$  are given by (10) and (11) respectively.
- iv)  $P_3(x^*, y^*, E^*)$  is the non-trivial interior equilibrium point for the exploited system where  $x^*$ ,  $y^*$  and  $E^*$  are given by (13), (12) and (14) respectively.

For existence of the non-trivial equilibrium points of the system (6) we have the following two theorems.

**Theorem-3:** For the unexploited system the non-trivial equilibrium point  $P_2(x_1, y_1, 0)$  exists if and only if  $\beta > s$  i.e. reproductive rate of the predator is greater than its mortality rate.

**Theorem-4:** For the exploited system the non-trivial interior equilibrium point  $P_3(x^*, y^*, E^*)$  exists if and only if

$$0 < \tau < p - \frac{\alpha\beta(abc+d)}{abqr(\beta-s)}.$$

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## V. LOCAL STABILITY ANALYSIS OF EQUILIBRIUM POINTS

The variational matrix of the system of equations (6) is  $V(x, y, E) = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix}$ ,  
where  $J_{11} = \frac{\partial}{\partial x} \left( \frac{dx}{dt} \right) = r - \frac{2rx}{k} - \alpha y$ ,  $J_{12} = \frac{\partial}{\partial y} \left( \frac{dx}{dt} \right) = -\alpha x$ ,  $J_{13} = \frac{\partial}{\partial E} \left( \frac{dx}{dt} \right) = 0$ ,  $J_{21} = \frac{\partial}{\partial x} \left( \frac{dy}{dt} \right) = \frac{\beta y}{k}$ ,  
 $J_{22} = \frac{\partial}{\partial y} \left( \frac{dy}{dt} \right) = \frac{\beta x}{k} - s - qE$ ,  $J_{23} = \frac{\partial}{\partial E} \left( \frac{dy}{dt} \right) = -qy$ ,  $J_{31} = \frac{\partial}{\partial x} \left( \frac{dE}{dt} \right) = 0$ ,  $J_{32} = \frac{\partial}{\partial y} \left( \frac{dE}{dt} \right) = abq(p - \tau)E$ ,  
 $J_{33} = \frac{\partial}{\partial E} \left( \frac{dE}{dt} \right) = ab\{q(q(p - \tau)y - c) - d\}$ .

$$\text{So } V(x, y, E) = \begin{pmatrix} r - \frac{2rx}{k} - \alpha y & -\alpha x & 0 \\ \frac{\beta y}{k} & \frac{\beta x}{k} - s - qE & -qy \\ 0 & abq(p - \tau)E & ab\{q(q(p - \tau)y - c) - d\} \end{pmatrix}. \quad (18)$$

$$\text{Now, } V(0,0,0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -(abc + d) \end{pmatrix} \text{ by (18).}$$

The eigen values of  $V(0,0,0)$  are  $r(> 0)$ ,  $-s(< 0)$  and  $-(abc + d)(< 0)$ . So the trivial equilibrium point  $P_0(0,0,0)$  of the system is unstable.

$$\text{Again, } V(k,0,0) = \begin{pmatrix} -r & -\alpha k & 0 \\ 0 & \beta - s & 0 \\ 0 & 0 & -(abc + d) \end{pmatrix} \text{ by (18).}$$

The eigen values of  $V(k,0,0)$  are  $-r(< 0)$ ,  $\beta - s$  and  $-(abc + d)(< 0)$ . So the boundary equilibrium point  $P_1(k,0,0)$  is stable or unstable according as  $\beta < s$  or  $\beta > s$ .

Therefore, if the reproductive rate of the predator is less than its death rate, then the boundary equilibrium of the system is asymptotically stable. Biologically it true that if the reproductive rate of the predator is less than its death rate then the predator species exponentially dies out and so the steady state of the prey species is equal to the environmental carrying capacity since the prey species is not harvested. If the reproductive rate of the predator species is greater than its death rate then the boundary equilibrium point of the system is unstable.

For the unexploited system ( $E = 0$ ), the variational matrix

$$V(x_1, y_1) = \begin{pmatrix} -\frac{rs}{\beta} & -\frac{rks}{\beta} \\ \frac{r}{k\alpha}(\beta - s) & 0 \end{pmatrix}.$$

$$\text{The characteristic equation of } V(x_1, y_1) \text{ is } \begin{vmatrix} -\frac{rs}{\beta} - \mu & -\frac{rks}{\beta} \\ \frac{r}{k\alpha}(\beta - s) & -\mu \end{vmatrix} = 0.$$

$$\text{i.e. } \mu^2 + \frac{rs}{\beta}\mu + \frac{r^2s(\beta-s)}{\alpha\beta} = 0 \quad (19)$$

Therefore, the sum of the eigen values is  $-\frac{rs}{\beta} < 0$  and product of the eigen values is  $\frac{r^2s(\beta-s)}{\alpha\beta}$ .

For existence of the non trivial steady state of the predator species it is assumed that the reproductive rate of the predator species is greater than its death rate i.e.  $\beta > s$ . So the product of the eigen values of  $V(x_1, y_1)$  is  $\frac{r^2s(\beta-s)}{\alpha\beta} > 0$ .

Therefore, the roots of the equation (19) are both real negative or complex conjugate with negative real parts.

The discriminant of the quadratic equation (19) is  $\frac{r^2s^2}{\beta^2} \left\{ 1 - \frac{4r^2(\beta-s)^2}{\alpha^2} \right\} < 0$  since the intrinsic growth rate of the prey species is greater than its catchability rate ( $r > \alpha$ ).



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Therefore, the eigen values of  $V(x_1, y_1)$  are complex conjugate with negative real parts. So the non trivial planar equilibrium point  $P_3(x_1, y_1, 0)$  for the unexploited system is a stable focus.

$$\text{Now, } V(x^*, y^*, E^*) = \begin{pmatrix} -\frac{rx^*}{k} & -\alpha x^* & 0 \\ \frac{\beta y^*}{k} & 0 & -qy^* \\ 0 & abq(p-\tau)E^* & 0 \end{pmatrix} \text{ by (18).}$$

The characteristic equation of the matrix  $V(x^*, y^*, E^*)$  is

$$\begin{vmatrix} -\frac{rx^*}{k} - \mu & -\alpha x^* & 0 \\ \frac{\beta y^*}{k} & -\mu & -qy^* \\ 0 & abq(p-\tau)E^* & -\mu \end{vmatrix} = 0$$

$$\text{or, } \mu^3 + \frac{rx^*}{k}\mu^2 + \left\{abq^2(p-\tau)E^*y^* + \frac{\alpha\beta x^*y^*}{k}\right\}\mu + \frac{1}{k}abrq^2(p-\tau)E^*x^*y^* = 0$$

$$\text{or, } \mu^3 + m_1\mu^2 + m_2\mu + m_3 = 0,$$

$$\text{where } m_1 = \frac{rx^*}{k} > 0, m_2 = abq^2(p-\tau)E^*y^* + \frac{\alpha\beta x^*y^*}{k} > 0 \text{ and}$$

$$m_3 = \frac{1}{k}abrq^2(p-\tau)E^*x^*y^* > 0.$$

$$\text{Now, } m_1m_2 - m_3 = \frac{1}{k}abrq^2(p-\tau)E^*x^*y^* + \frac{1}{k^2}r\alpha\beta x^{*2}y^* - \frac{1}{k}abrq^2(p-\tau)E^*x^*y^* \\ = \frac{1}{k^2}r\alpha\beta x^{*2}y^* > 0.$$

Therefore, by Routh-Hurwitz criterion [25] the non-trivial interior equilibrium point of the system (6) is always locally asymptotically stable if it exists.

## VI. GLOBAL STABILITY ANALYSIS FOR THE NON-TRIVIAL EQUILIBRIUM POINTS

We now discuss the global stability of the non-trivial equilibrium points of the system (6) separately for unexploited and exploited systems.

### Case-I: For the unexploited system ( $E = 0$ ).

We now examine the global stability of the non-trivial equilibrium point  $P_2(x_1, y_1, 0)$  of the system (6) when  $E = 0$  in the Lyapunov sense [26]. Since the environmental carrying capacity for the populations is fixed and the predator density is low, it is assumed that the solutions of the dynamical system (6) for unexploited fishery i.e. when  $E = 0$  are eventually uniformly bounded in the finite region in x-y plane.

We define a Lyapunov function as

$$L(x, y) = x - x_1 - x_1 \ln\left(\frac{x}{x_1}\right) + \lambda \left\{ y - y_1 - y_1 \ln\left(\frac{y}{y_1}\right) \right\},$$

where  $\lambda$  is a suitable positive constant to be determined in the subsequent steps.

$$\text{Here } L(x_1, y_1) = 0 \text{ and } \lim_{(x,y) \rightarrow (0,0)} L(x, y) = \lim_{(x,y) \rightarrow (\infty, \infty)} L(x, y) = \infty.$$

The time derivative of  $L(x, y)$  along the solution of (6) is

$$\begin{aligned} \frac{d}{dt} \{L(x, y)\} &= \left(\frac{x-x_1}{x}\right) \frac{dx}{dt} + \lambda \left(\frac{y-y_1}{y}\right) \frac{dy}{dt} \\ &= (x-x_1) \left\{ r \left(1 - \frac{x}{k}\right) - \alpha y \right\} + \lambda (y-y_1) \left\{ \left(\frac{\beta x}{k} - s\right) \right\} \text{ by (6), since } E = 0. \\ &= (x-x_1) \left[ \left\{ r \left(1 - \frac{x}{k}\right) - \alpha y \right\} - \left\{ r \left(1 - \frac{x_1}{k}\right) - \alpha y_1 \right\} \right] + \lambda (y-y_1) \left\{ \left(\frac{\beta x}{k} - s\right) - \left(\frac{\beta x_1}{k} - s\right) \right\} \\ &= (x-x_1) \left\{ -\frac{r}{k}(x-x_1) - \alpha(y-y_1) \right\} + \frac{\lambda\beta}{k} (x-x_1)(y-y_1) \end{aligned}$$

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$$= -\frac{r}{k}(x-x_1)^2 - \alpha(x-x_1)(y-y_1) + \frac{\lambda\beta}{k}(x-x_1)(y-y_1)$$

$$= -\frac{r}{k}(x-x_1)^2 < 0 \text{ for } \lambda = \frac{k\alpha}{\beta} > 0.$$

Therefore,  $\frac{d}{dt}\{L(x, y)\} < 0 \forall (x, y) \in R_2^+$  where  $R_2^+ = \{(x, y): x, y \in R \text{ and } x > 0, y > 0\}$   
and  $\frac{d}{dt}\{L(x_1, y_1)\} = 0$ .

Hence by Lassel's invariance principle [27], the non-trivial equilibrium point  $P_2(x_1, y_1, 0)$  for the system (6) when  $E = 0$  is globally asymptotically stable in the positive quadrant of x-y plane.

## Case-II: For the exploited system ( $E \neq 0$ ).

Here we also examine the global stability of the non-trivial interior equilibrium point  $P_3(x^*, y^*, E^*)$  of the system (6) in the Lyapunov sense [26].

For the fixed environmental carrying capacity for the populations and also predator density being low, the prey and predator densities are bounded in the finite region. Since the regulatory agencies control the over exploitation by imposing the suitable tax, so the effort level is also bounded. Thus the solutions of the dynamical system (6) are uniformly bounded in the finite region  $R_3^+ = \{(x, y, E): x, y, E \in R, x > 0, y > 0, E > 0\}$ .

Now, we consider the Lypunov function as follows:

$L(x, y, E) = x - x^* - x^* \ln\left(\frac{x}{x^*}\right) + L_1 \left\{y - y^* - y^* \ln\left(\frac{y}{y^*}\right)\right\} + L_2 \left\{E - E^* - E^* \ln\left(\frac{E}{E^*}\right)\right\}$  where  $L_1$  and  $L_2$  are suitable positive constants to be determined in the subsequent steps.

Here  $L(x^*, y^*, E^*) = 0$  and  $\lim_{(x,y,E) \rightarrow (0,0,0)} L(x, y, E) = \lim_{(x,y,E) \rightarrow (\infty, \infty, \infty)} L(x, y, E) = \infty$ .

The time derivative of  $L(x, y, E)$  along the solution of (6) is

$$\begin{aligned} \frac{d}{dt}\{L(x, y, E)\} &= \left(\frac{x-x^*}{x}\right) \frac{dx}{dt} + L_1 \left(\frac{y-y^*}{y}\right) \frac{dy}{dt} + L_2 \left(\frac{E-E^*}{E}\right) \frac{dE}{dt} \\ &= (x-x^*) \left\{r \left(1 - \frac{x}{k}\right) - \alpha y\right\} + L_1 (y-y^*) \left\{\left(\frac{\beta x}{k} - s - qE\right) + L_2 (E-E^*) [ab\{q(p-\tau)y - c\} - d]\right\} \\ &= (x-x^*) \left\{\left[r \left(1 - \frac{x}{k}\right) - \alpha y\right] - \left[r \left(1 - \frac{x^*}{k}\right) - \alpha y^*\right]\right\} + L_1 (y-y^*) \left\{\left(\frac{\beta x}{k} - s - qE\right) - \left(\frac{\beta x^*}{k} - s - qE^*\right)\right\} \\ &\quad + L_2 (E-E^*) [ab\{q(p-\tau)y - c\} - d] - [ab\{q(p-\tau)y^* - c\} - d] \text{ by (7).} \\ &= (x-x^*) \left\{-\frac{r}{k}(x-x^*) - \alpha(y-y^*)\right\} + L_1 (y-y^*) \left\{\frac{\beta}{k}(x-x^*) - q(E-E^*)\right\} + L_2 abq(p-\tau)(y-y^*)(E-E^*) \\ &= -\frac{r}{k}(x-x^*)^2 - \alpha(x-x^*)(y-y^*) + L_1 \frac{\beta}{k}(x-x^*)(y-y^*) \\ &\quad - L_1 q(y-y^*)(E-E^*) + L_2 abq(p-\tau)(E-E^*)(y-y^*) \\ &= -\frac{r}{k}(x-x^*)^2 < 0 \text{ for } L_1 = \frac{k\alpha}{\beta} > 0 \text{ and } L_2 = \frac{k\alpha}{ab\beta(p-\tau)} > 0. \end{aligned}$$

Therefore,  $\frac{d}{dt}\{L(x, y, E)\} < 0 \forall (x, y, E) \in R_3^+$

where  $R_3^+ = \{(x, y, E): x, y, E \in R \text{ and } x > 0, y > 0, E > 0\}$  and  $\frac{d}{dt}\{L(x^*, y^*, E^*)\} = 0$ .

Hence by Lassel's invariance principle [12], the non-trivial interior equilibrium point  $P_3(x^*, y^*, E^*)$  for the system (6) is globally asymptotically stable in  $R_3^+$ .

## VII. NUMERICAL EXAMPLES

### Example 1: (Conservative prey predator system)

Let  $r = 10, k = 100, \alpha = 10\%, \beta = 5, s = 30\%, q = 1\%, a = 0.7, b = 0.8, d = 20\%, c = 20$  and  $p = 40$  in appropriate units.

Here  $\frac{r}{\alpha} = 100 = k$ , so by theorem-1, the system is a conservative system i.e. the sum of the steady states of the prey and predator species is always constant and equal to the environmental carrying capacity.

$P_1(100, 0, 0)$  is the axial equilibrium point of the system. Since  $\beta > s$ , so by theorem-3, the planar equilibrium  $P_2(x_1, y_1, 0)$  of the system exists.



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Here  $x_1 = \frac{ks}{\beta} = 6$  and  $y_1 = \frac{r}{\alpha} \left(1 - \frac{s}{\beta}\right) = 94$ . So  $P_2(6,94,0)$  is the planar equilibrium of the system which is a stable focus. In this case sum of the steady states of the prey and predator species is 100 and it is equal to the environmental carrying capacity. So theorem-1 is verified for the unexploited system.

From this result it is clear that for the unexploited system the steady state of the prey population is very low whereas for the predator population it is very high and it is very close to the environmental carrying capacity. Biologically it is true that if the predator species is not harvested then the prey species gradually decreases and finally extinct due to predation effect. So harvesting of the predator species is an important role for the balanced ecosystem.

Now,  $p - \frac{\alpha\beta(abc+d)}{abqr(\beta-s)} = 18.34$ . So in order to exist the non trivial interior equilibrium point of the system the regulatory agencies should impose the tax  $\tau$  such that  $0 < \tau < 18.34$ , by theorem-4.

For different suitable taxes the different non-trivial interior equilibrium points are given below.

Table-1: Steady states for different taxes

Tax $\tau$	$x^*$	$y^*$	$E^*$	$x^* + y^*$
0	49.107	50.893	215.536	100
1	47.802	52.198	209.011	100
2	46.429	53.571	202.143	100
3	44.981	55.019	194.903	100
4	43.452	56.548	187.262	100
5	41.837	58.163	179.184	100
6	40.126	59.874	170.630	100
7	38.312	61.688	161.558	100
8	36.384	63.616	151.920	100
9	34.332	65.668	141.659	100
10	32.143	67.857	130.714	100
11	29.803	70.197	119.015	100
12	27.296	72.704	106.480	100
13	24.603	75.397	93.016	100
14	21.703	78.297	78.516	100
15	18.571	81.429	62.857	100
16	15.179	84.821	45.893	100
17	11.491	88.509	27.453	100
18	7.468	92.532	7.338	100
18.34	6.000	94.000	0	100

From table-1 and figure-1 it is clear that when the tax  $\tau$  is increased then the effort level is decreased, the steady state level of predator is increased and the steady state level of prey is decreased. Since only the predator species is harvested so it true that if effort is decreased then the predator density is increased. Again since the predator species kill prey species so increase of predator population implies decrease of prey population. From the above table and figure it is also seen that if the regulatory agencies impose no tax then the steady state levels of both the prey and predator populations are nearly equal and the harvesting effort level is maximum. When the regulatory agencies impose maximum tax ( $\tau = 18.34$ ) then the effort level is zero and the non trivial steady state becomes the non trivial planar equilibrium  $P_2(6,94,0)$ . It is also seen that whatever the tax be imposed by the regulatory agencies, the sum of the steady states of the prey and predator populations is always constant and equal to the environmental carrying capacity 100. So it is a conservative prey predator exploited system. Therefore, theorem-1 is also verified for the exploited system.

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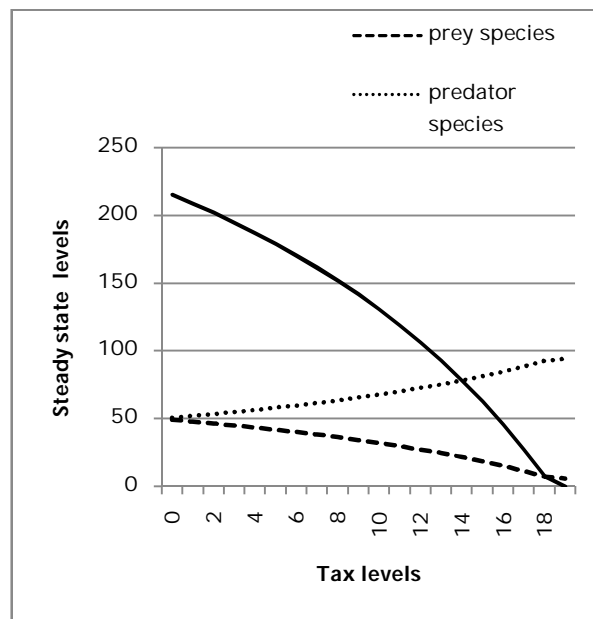


Figure-1: Steady states  $(x^*, y^*, E^*)$  for different taxes

## Example 2: (Non-conservative prey predator system)

Let  $r = 10, k = 100, \alpha = 15\%, \beta = 5, s = 30\%, q = 1\%, a = 0.7, b = 0.8, d = 20\%, c = 20$  and  $p = 40$  in appropriate units.

Here  $\frac{r}{\alpha} = 68.67 < k (= 100)$ , so by theorem-2, the system is a non-conservative system i.e. the sum of the steady states of the prey and predator species is always less than the environmental carrying capacity.

Since  $\beta > s$ , so by theorem-3, the planar equilibrium  $P_2(x_1, y_1, 0)$  of the system exists.

Here  $x_1 = \frac{ks}{\beta} = 6$  and  $y_1 = \frac{r}{\alpha} \left(1 - \frac{s}{\beta}\right) = 62.67$ . So  $P_2(6, 62.67, 0)$  is the planar equilibrium of the system which is a stable focus. In this case sum of the steady states of the prey and predator species is  $68.67 < k (= 100)$ . So theorem-1 is verified for the unexploited system.

Now,  $p - \frac{\alpha\beta(abc+d)}{abqr(\beta-s)} = 7.516$ . So in order to exist the non trivial interior equilibrium point of the system the regulatory agencies should impose the tax  $\tau$  such that  $0 < \tau < 7.516$ , by theorem-4.

For different suitable taxes the different non-trivial interior equilibrium points are given below.

Table-2: Steady states for different taxes

Tax $\tau$	$x^*$	$y^*$	$E^*$	$x^* + y^*$
0	23.661	50.893	88.304	74.554
1	21.703	52.198	78.516	73.901
2	19.643	53.571	68.214	73.214
3	17.471	55.019	57.355	72.490
4	15.179	56.548	45.893	71.727
5	12.756	58.163	33.776	70.919
6	10.189	59.874	20.946	70.063
7	7.468	61.689	7.338	69.157
7.516	6	62.669	0.012	68.669

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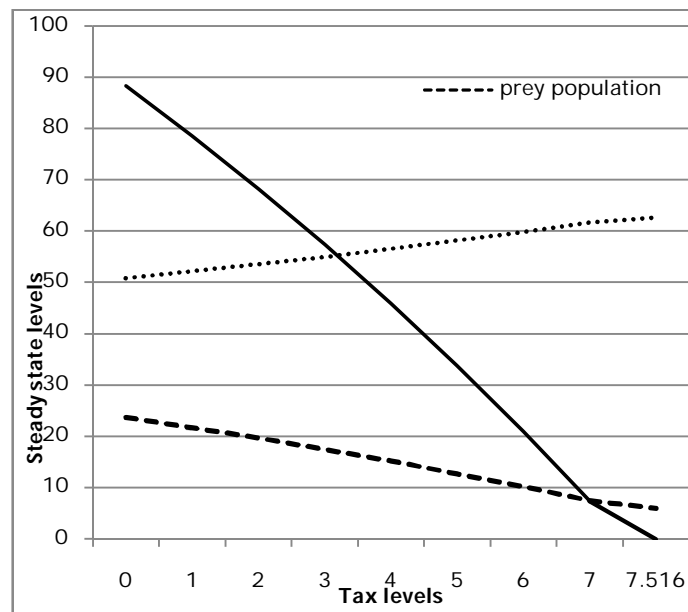


Figure-2: Steady states  $(x^*, y^*, E^*)$  for different taxes

From table-2 and figure-2 it is also clear that when the tax  $\tau$  is increased then the effort level is decreased, the steady state level of predator is increased and the steady state level of prey is decreased. From the above table and figure it is also seen that if the regulatory agencies impose no tax then the steady state level of the effort is maximum, steady state level of prey species is also maximum and the steady state level of predator species is minimum. When the regulatory agencies impose maximum tax ( $\tau = 7.516$ ) then the effort level is very low, near to zero and the non trivial steady state becomes the non trivial planar equilibrium  $P_2(6,66.67,0)$ . It is also seen that whatever the tax be imposed by the regulatory agencies, the sum of the steady states of the prey and predator populations is not constant but always less than the environmental carrying capacity 100. So it is a non-conservative prey predator exploited system. Therefore, the theorem-2 is also verified for the exploited system.

## VIII. CONCLUSION

This model is suitable for a fishery having a prey-predator community in which the prey density is high and that of the predator species is low. An appropriate example might be a krill-whale fishery. The important result of this model is that when the ratio of the intrinsic growth rate of the prey species and the removal rate of the prey by the predator species is equal to the environmental carrying capacity, then the sum of the population densities of the prey and predator species is always constant and equal to the carrying capacity, whatever the tax be imposed by the regulatory agencies. So whole the fish populations in the environment are always saturated whatever the harvest may be. If the non-trivial equilibrium points exist then these points are always locally and globally stable in this model.

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